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THE TENSOR EQUATION $AX + XA = G$

MIKE SCHEIDLER

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We study the second-order tensor equation $AX + XA = G$ for symmetric, positive-definite A and arbitrary G. Motivated by applications in the continuum mechanics literature, we also examine some special cases where G depends on A and another tensor H. For arbitrary dimensions, we establish relations between the solutions X for various forms of G. These results, together with Rivlin's identities for tensor polynomials in two variables, are applied in two and three dimensions to obtain explicit formulas for X in direct (component-free) notation. The results include new formulas as well as new derivations of previously known formulas. An application to the kinematics of rigid motions is considered.

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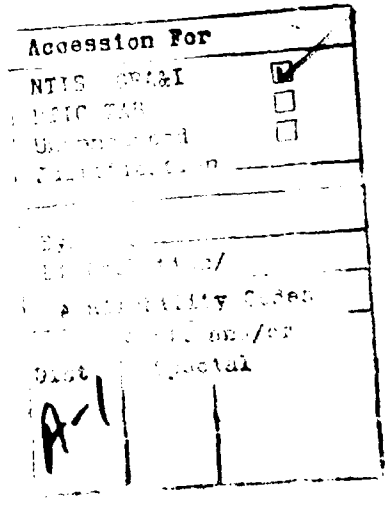
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1. INTRODUCTION

Tensor equations of the form

$$\boldsymbol{A}\boldsymbol{X} + \boldsymbol{X}\boldsymbol{A} = \boldsymbol{G} \tag{1.1}$$

arise in a variety of problems in continuum mechanics; cf. Sidoroff (1978), Guo (1984), Hoger and Carlson (1984), Leonov (1976), Stickforth (1982–1983), Mehrabadi and Nemat-Nasser (1987), Dienes (1979), Stickforth and Wegener (1988), Hoger (1986), Nemat-Nasser (1990), Truesdell (in press), Wheeler (1990), and Scheidler (to be published) for applications to kinematics and constitutive modeling. Here the second-order tensors $\boldsymbol{A}$ and $\boldsymbol{G}$ are regarded as known, and we wish to solve (1.1) for the second-order tensor $\boldsymbol{X}$. For the applications considered in the aforementioned references, $\boldsymbol{A}$ is symmetric and positive-definite. This restriction, which will be imposed here also, guarantees that a solution $\boldsymbol{X}$ of (1.1) exists and is unique. Indeed, relative to any principal basis for $\boldsymbol{A}$, this solution is given by the simple component formula

$$X_{ij} = \frac{G_{ij}}{a_i + a_j}\,, \tag{1.2}$$

where a_i are the (necessarily positive) eigenvalues of $\boldsymbol{A}$. Of course, to actually compute the components of $\boldsymbol{X}$ by means of (1.2), we must first solve for the eigenvalues and eigenvectors of $\boldsymbol{A}$. This has motivated several authors to seek explicit formulas for the tensor $\boldsymbol{X}$ directly in terms of the tensors $\boldsymbol{A}$ and $\boldsymbol{G}$. An example is the formula of Sidoroff (1978) and Guo (1984) for the case where $\boldsymbol{G}$ is skew and the underlying vector space is three-dimensional:

$$\boldsymbol{X} = \frac{1}{\mathrm{I}_{\boldsymbol{A}}\,\mathrm{II}_{\boldsymbol{A}} - \mathrm{III}_{\boldsymbol{A}}}\left[(\mathrm{I}_{\boldsymbol{A}}^2 - \mathrm{II}_{\boldsymbol{A}})\,\boldsymbol{G} - (\boldsymbol{A}^2\boldsymbol{G} + \boldsymbol{G}\boldsymbol{A}^2)\right]. \tag{1.3}$$

Here $\mathrm{I}_{\boldsymbol{A}}$, $\mathrm{II}_{\boldsymbol{A}}$, and $\mathrm{III}_{\boldsymbol{A}}$ denote the principal invariants of $\boldsymbol{A}$. Since equations of the form (1.3) are often said to be displayed in direct notation, we will refer to formulas of this type as *direct formulas* for $\boldsymbol{X}$ or *direct solutions of* (1.1).

Although the derivation of the component formula (1.2) is a trivial exercise, the derivation of direct formulas for $\boldsymbol{X}$ is a nontrivial problem. Furthermore, whereas the component formula (1.2) is valid for any finite-dimensional vector space, the direct formulas become more complex as the

dimension of the underlying vector space increases. For example, in the two-dimensional case the solution of (1.1) for skew $\boldsymbol{G}$ is

$$\boldsymbol{X} = \frac{1}{\mathrm{I}_A}\,\boldsymbol{G}\ , \tag{1.4}$$

which is substantially simpler than its three-dimensional counterpart (1.3). The formula (1.4) is a special case of the second of two direct formulas obtained by Hoger and Carlson (1984) for arbitrary $\boldsymbol{G}$ in the two-dimensional case. For the three-dimensional case, Sidoroff (1978), Hoger and Carlson (1984), Leonov (1976) and Stickforth (1982–1983) have obtained direct solutions of (1.1) for arbitrary $\boldsymbol{G}$. For the two-dimensional and three-dimensional cases, Mehrabadi and Nemat-Nasser (1987) obtained direct formulas for $\boldsymbol{X}$ when $\boldsymbol{G} = \boldsymbol{AH} - \boldsymbol{HA}$ with $\boldsymbol{H}$ arbitrary. For the three-dimensional case with $\boldsymbol{G}$ (and thus $\boldsymbol{X}$) skew, Sidoroff (1978), Guo (1984) and Dienes (1979) obtained formulas for the axial vector* of $\boldsymbol{X}$ in terms of $\boldsymbol{A}$ and the axial vector of $\boldsymbol{G}$.

The direct solutions of (1.1) have been used to obtain direct formulas for various tensors of interest in continuum mechanics. Examples include formulas for:

(1) the material time derivatives of the stretch and rotation tensors in terms of either the velocity gradient (Guo 1984; Stickforth and Wegener 1988) or the stretching and spin tensors (Mehrabadi and Nemat-Nasser 1987; Dienes 1979; Stickforth and Wegener 1988; Hoger 1986) or the material time derivatives of the Cauchy-Green tensors (Hoger and Carlson 1984);

(2) the stretching tensor in terms of the Jaumann rate of the left Cauchy-Green tensor (Sidoroff 1978; Leonov 1976);

(3) the derivative of the square root of a tensor (Hoger and Carlson 1984);

(4) the elastic and plastic spin tensors in terms of the corresponding stretching tensors (Nemat-Nasser 1990);

*The *axial vector* of a skew tensor G is the unique vector g such that $\mathbf{Gv} = \mathbf{g} \times \mathbf{v}$ for every vector v, where $\mathbf{g} \times \mathbf{v}$ denotes the vector cross product of g and v.

(5) the spin tensor in terms of the rotational momentum and the Euler tensor, for a body undergoing a rigid motion (Truesdell, in press).

The derivatives of the stretch and rotation tensors with respect to the deformation gradient (Wheeler 1990; Scheidler, to be published) also satisfy tensor equations of the form (1.1).

Now the tensor G on the right-hand side of (1.1) may depend on A and some other tensor, say H, which is regarded as known. Expressions for G of the form

$$\begin{aligned} &AH - HA,\ AHA,\ HA - AH^T,\ AH - H^TA, \\ &AH + H^TA\ \ HA + AH^T,\ A^2H + HA^2 \end{aligned} \tag{1.5}$$

occur in the applications listed above; here H^T denotes the transpose of H. Of course, to obtain a direct solution of (1.1) in these cases we can simply substitute the expression for G in terms of A and H into one of the direct solutions which are valid for any G. However, in many of these cases there exist direct formulas for X which are simpler than (but necessarily equivalent to) the formulas obtained by this approach. One of the goals of this paper is to develop methods which yield these simpler formulas with a minimal amount of computational effort.

The paper is organized as follows. Section 2 contains some preliminary material. In Sections 3 and 4 we study the general properties of the solutions of the equation (1.1) and the equation obtained from (1.1) by setting $G = AH - HA$, i.e., the tensor equation

$$AX + XA = AH - HA\ . \tag{1.6}$$

By "general" properties we mean properties which are independent of the dimension of the underlying vector space. We show that when G is given in terms of A and some tensor H by one of the expressions in (1.5), the solution of (1.1) has a simple expression in terms of the solution of (1.6). Furthermore, we show that the solution of (1.1) for arbitrary G can be expressed in terms of the solution of (1.6) with $H = G$.

In Sections 5 and 6 we restrict attention to the three-dimensional and two-dimensional cases, respectively, since these are the cases of most interest in continuum mechanics. We utilize

Rivlin's (1955) identities for tensor polynomials in two variables to derive direct solutions of (1.1) for the special case where $\boldsymbol{G}$ is orthogonal to every polynomial in $\boldsymbol{A}$. These formulas are then used to derive direct solutions of (1.6) for arbitrary $\boldsymbol{H}$, and those formulas are in turn used to derive direct solutions of (1.1) for arbitrary $\boldsymbol{G}$. When $\boldsymbol{G}$ is given by one of the expressions in (1.5), simpler direct solutions of (1.1) follow from the identities in Section 4 and our direct solutions of (1.6). Several of the direct formulas derived in this paper have been obtained previously by some of the authors mentioned above. However, our derivations differ from theirs, and we derive some new formulas as well. Our emphasis on the special case (1.6) of the tensor equation (1.1) was motivated by some results in the paper of Mehrabadi and Nemat-Nasser (1987); see the comments following equation (4.5) in this paper.

In Section 7 we use one of our formulas to derive a new direct formula for the spin tensor in terms of the rotational momentum and the tensor of inertia for a body undergoing a rigid motion. In a follow-up paper (Scheidler, to be published) we use the general properties and the direct formulas obtained here to derive direct formulas for the derivatives of the stretch and rotation tensors with respect to the deformation gradient. We apply those results to the derivation of direct formulas for the material time derivatives of the stretch and rotation tensors and also to the derivation of direct formulas for the stress in a hyperelastic material in terms of the derivative of the strain energy with respect to the right stretch tensor. The latter problem was solved recently by Wheeler (1990) for the isotropic case.

2. ALGEBRAIC PRELIMINARIES

The underlying vector space is a finite-dimensional inner product space $\mathcal{V}$ over the reals. By a second-order tensor we mean a linear transformation from $\mathcal{V}$ into $\mathcal{V}$. The vector space of all second-order tensors is denoted by Lin. Sym and Skw are the subspaces of Lin consisting of symmetric and skew tensors. Throughout this paper A denotes an arbitrary symmetric positive- definite tensor with eigenvalues $a_i > 0$, i = 1, 2, ..., N, where N is the dimension of $\mathcal{V}$. The identity tensor is denoted by $\boldsymbol{I}$; $\boldsymbol{B}$, $\boldsymbol{G}$, $\boldsymbol{H}$, and $\boldsymbol{X}$ denote second-order tensors which are arbitrary unless specified otherwise. The inner product of $\boldsymbol{B}$ and $\boldsymbol{H}$ is

$$\boldsymbol{B} \cdot \boldsymbol{H} := \mathrm{tr}\,(\boldsymbol{B}^{\mathrm{T}}\boldsymbol{H})\,. \tag{2.1}$$

$$\text{sym } \boldsymbol{H} := \frac{1}{2}\,(\boldsymbol{H} + \boldsymbol{H}^T)\,, \qquad \text{skw } \boldsymbol{H} := \frac{1}{2}\,(\boldsymbol{H} - \boldsymbol{H}^T)\,. \tag{2.2}$$

We record the following identities for later use:

$$\begin{aligned} 2\,\text{sym}\,(\boldsymbol{AH}) &= \boldsymbol{AH} + \boldsymbol{H}^T\boldsymbol{A} = \boldsymbol{AH} + \boldsymbol{HA} - 2\,(\text{skw}\,\boldsymbol{H})\,\boldsymbol{A} \\ &= \boldsymbol{A}\,\text{sym}\,\boldsymbol{H} + (\text{sym}\,\boldsymbol{H})\,\boldsymbol{A} + \boldsymbol{A}\,\text{skw}\,\boldsymbol{H} - (\text{skw}\,\boldsymbol{H})\,\boldsymbol{A}\,, \end{aligned} \tag{2.3}$$

$$\begin{aligned} 2\,\text{sym}\,(\boldsymbol{HA}) &= \boldsymbol{HA} + \boldsymbol{AH}^T = \boldsymbol{AH} + \boldsymbol{HA} - 2\boldsymbol{A}\,\text{skw}\,\boldsymbol{H} \\ &= \boldsymbol{A}\,\text{sym}\,\boldsymbol{H} + (\text{sym}\,\boldsymbol{H})\,\boldsymbol{A} + (\text{skw}\,\boldsymbol{H})\,\boldsymbol{A} - \boldsymbol{A}\,\text{skw}\,\boldsymbol{H}\,, \end{aligned} \tag{2.4}$$

$$\begin{aligned} 2\,\text{skw}\,(\boldsymbol{AH}) &= \boldsymbol{AH} - \boldsymbol{H}^T\boldsymbol{A} = \boldsymbol{AH} - \boldsymbol{HA} + 2\,(\text{skw}\,\boldsymbol{H})\,\boldsymbol{A} \\ &= \boldsymbol{A}\,\text{sym}\,\boldsymbol{H} - (\text{sym}\,\boldsymbol{H})\,\boldsymbol{A} + \boldsymbol{A}\,\text{skw}\,\boldsymbol{H} + (\text{skw}\,\boldsymbol{H})\,\boldsymbol{A}\,, \end{aligned} \tag{2.5}$$

$$\begin{aligned} 2\,\text{skw}\,(\boldsymbol{HA}) &= \boldsymbol{HA} - \boldsymbol{AH}^T = \boldsymbol{HA} - \boldsymbol{AH} + 2\,\boldsymbol{A}\,\text{skw}\,\boldsymbol{H} \\ &= (\text{sym}\,\boldsymbol{H})\,\boldsymbol{A} - \boldsymbol{A}\,\text{sym}\,\boldsymbol{H} + \boldsymbol{A}\,\text{skw}\,\boldsymbol{H} + (\text{skw}\,\boldsymbol{H})\,\boldsymbol{A}\,. \end{aligned} \tag{2.6}$$

A tensor $\boldsymbol{H}$ is a polynomial in $\boldsymbol{A}$ if $\boldsymbol{H} = c_m\,\boldsymbol{A}^m + c_{m-1}\,\boldsymbol{A}^{m-1} + \ldots + c_1\,\boldsymbol{A} + c_0\,\mathbf{I}$ for some nonnegative integer m and scalars $c_0, c_1, \ldots, c_m$. The subspace of Lin consisting of all polynomials in $\boldsymbol{A}$ is denoted by $\mathcal{P}(\boldsymbol{A})$. The Cayley-Hamilton theorem implies that

$$\begin{aligned} \mathcal{P}(\boldsymbol{A}) &= \text{span}\,\{\boldsymbol{I}, \boldsymbol{A}, \ldots, \boldsymbol{A}^{N-1}\} \\ &= \text{span}\,\{\boldsymbol{A}, \boldsymbol{A}^2, \ldots, \boldsymbol{A}^N\} \\ &= \text{span}\,\{\ldots, \boldsymbol{A}^{-2}, \boldsymbol{A}^{-1}, \boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}^2, \ldots\}\,. \end{aligned} \tag{2.7}$$

Let $\mathcal{P}(\boldsymbol{A})^\perp$ denote the orthogonal complement of $\mathcal{P}(\boldsymbol{A})$, i.e., $\boldsymbol{H} \in \mathcal{P}(\boldsymbol{A})^\perp$ iff $\boldsymbol{H}$ is orthogonal to every polynomial in $\boldsymbol{A}$. Then from (2.7) and (2.1) it follows that

$$H \in \mathcal{P}(A)^{\perp} \Leftrightarrow \text{tr}\,(A^k H) = 0, \quad \forall\, k \in \{0, 1, \ldots, N-1\}$$

$$\Leftrightarrow \text{tr}\,(A^k H) = 0, \quad \forall\, k \in \{1, 2, \ldots, N\}$$

$$\Leftrightarrow \text{tr}\,(A^k H) = 0, \quad \forall \text{ integer } k\,. \tag{2.8}$$

In particular, if $H \in \mathcal{P}(A)^{\perp}$ then H is deviatoric, i.e., tr $H = 0$. Also note that every skew tensor belongs to $\mathcal{P}(A)^{\perp}$.

By a fourth-order tensor we mean a linear transformation from Lin into Lin. $\mathbf{K}$ denotes an arbitrary fourth-order tensor; the image of $H \in$ Lin under $\mathbf{K}$ is denoted by $\mathbf{K}[H]$. It is easily verified that *the following two conditions are equivalent:*

$$\begin{aligned} &\mathbf{K}[H^T] = (\mathbf{K}[H])^T; \\ &\mathbf{K}[\text{sym } H] = \text{sym } \mathbf{K}[H] \quad \text{and} \quad \mathbf{K}[\text{skw } H] = \text{skw } \mathbf{K}[H]\,. \end{aligned} \tag{2.9}$$

If these conditions hold for each $H \in$ Lin we say that $\mathbf{K}$ is *even.* Similarly, it is easily verified that *the following two conditions are equivalent:*

$$\begin{aligned} &\mathbf{K}[H^T] = -(\mathbf{K}[H])^T; \\ &\mathbf{K}[\text{sym } H] = \text{skw } \mathbf{K}[H] \quad \text{and} \quad \mathbf{K}[\text{skw } H] = \text{sym } \mathbf{K}[H]\,. \end{aligned} \tag{2.10}$$

If these conditions hold for every $H \in$ Lin, we say that $\mathbf{K}$ is *odd.* The transpose of $\mathbf{K}$ is the unique fourth-order tensor $\mathbf{K}^T$ with the property

$$B \cdot \mathbf{K}^T[H] = \mathbf{K}[B] \cdot H\,, \quad \forall\, B, H \in \text{Lin}\,. \tag{2.11}$$

We say that $\mathbf{K}$ is symmetric if $\mathbf{K}^T = \mathbf{K}$.

3. THE TENSORS $\mathbf{L}_A$ AND $\mathbf{M}_A$

In this section we study the properties of five fourth-order tensors associated with any symmetric

positive-definite second-order tensor A. The two most important of these are denoted by $\mathbf{L}_A$ and $\mathbf{M}_A$ and characterized by the conditions

$$X = \mathbf{L}_A\,[G] \Leftrightarrow AX + XA = G\,, \tag{3.1}$$

$$X = \mathbf{M}_A\,[H] \Leftrightarrow AX + XA = AH - HA\,. \tag{3.2}$$

The other fourth-order tensors are denoted by $\mathbf{N}_A$, $\mathbf{B}_A$ and $\mathbf{C}_A$. $\mathbf{N}_A$ is introduced primarily to simplify the statement of some of the identities relating $\mathbf{L}_A$ and $\mathbf{M}_A$. $\mathbf{B}_A$ and $\mathbf{C}_A$ are introduced to facilitate the definition of the other tensors and the derivation of their properties.

The fourth-order tensor $\mathbf{B}_A$ is defined by

$$\mathbf{B}_A\,[X] := AX + XA\,. \tag{3.3}$$

Note that $\mathbf{B}_A$ is invertible. Indeed, the condition $\mathbf{B}_A\,[X] = G$ is just the tensor equation (1.1) which, as pointed out in the introduction, has a unique solution X for any given G. We denote the inverse of $\mathbf{B}_A$ by $\mathbf{L}_A$:

$$\mathbf{L}_A := (\mathbf{B}_A)^{-1}\,. \tag{3.4}$$

Then $\mathbf{B}_A\,[X] = G$ iff $X = \mathbf{L}_A\,[G]$, which is equivalent to the statement (3.1). If we let $\mathbf{I}$ denote the fourth-order identity tensor on Lin, then*

$$\mathbf{L}_A\,\mathbf{B}_A = \mathbf{B}_A\,\mathbf{L}_A = \mathbf{I}\,, \tag{3.5}$$

which is equivalent to the identities

$$\mathbf{L}_A\,[AH + HA] = A\,\mathbf{L}_A\,[H] + \mathbf{L}_A\,[H]\,A = H\,. \tag{3.6}$$

*The product of two fourth-order tensors is defined to be their composition: $(\mathbf{L}_A\,\mathbf{B}_A)\,[X] := \mathbf{L}_A\,[\mathbf{B}_A\,[X]]$ for each $X \in$ Lin.

The fourth-order tensor $\boldsymbol{C}_A$ is defined by

$$\boldsymbol{C}_A\,[H] := AH - HA\,. \tag{3.7}$$

It is easily verified that $\boldsymbol{C}_A$ and $\boldsymbol{B}_A$ commute; indeed,

$$\boldsymbol{C}_A\,\boldsymbol{B}_A = \boldsymbol{B}_A\,\boldsymbol{C}_A = \boldsymbol{C}_{A^2}\,. \tag{3.8}$$

We define the fourth-order tensor $\boldsymbol{M}_A$ by

$$\boldsymbol{L}_A\,\boldsymbol{C}_A = \boldsymbol{C}_A\,\boldsymbol{L}_A\,, \tag{3.9}$$

where $(3.9)_2$ follows from pre-multiplying and post-multiplying $(3.8)_1$ by $\boldsymbol{L}_A$ and using (3.5). From (3.9) and (3.7) we see that

$$\boldsymbol{M}_A\,[H] = \boldsymbol{L}_A\,[AH - HA] = A\,\boldsymbol{L}_A\,[H] - \boldsymbol{L}_A\,[H]\,A\,. \tag{3.10}$$

Then (3.2) follows from (3.1) (with $G = AH - HA$) and $(3.10)_1$. From (3.9), (3.8), and (3.5) we obtain

$$\boldsymbol{M}_A\,\boldsymbol{B}_A = \boldsymbol{B}_A\,\boldsymbol{M}_A = \boldsymbol{C}_A = \boldsymbol{L}_A\,\boldsymbol{C}_{A^2} = \boldsymbol{C}_{A^2}\,\boldsymbol{L}_A\,, \tag{3.11}$$

which is equivalent to the identities

$$\begin{aligned}\boldsymbol{M}_A\,[AH + HA] &= A\,\boldsymbol{M}_A\,[H] + \boldsymbol{M}_A\,[H]\,A\\ &= AH - HA\\ &= \boldsymbol{L}_A\,[A^2H - HA^2] = A^2\,\boldsymbol{L}_A\,[H] - \boldsymbol{L}_A\,[H]\,A^2\,.\end{aligned} \tag{3.12}$$

The fourth-order tensor $\boldsymbol{N}_A$ is defined by

$$\boldsymbol{N}_A := \boldsymbol{M}_A\,\boldsymbol{C}_A = \boldsymbol{C}_A\,\boldsymbol{M}_A = \boldsymbol{L}_A\,(\boldsymbol{C}_A)^2\,, \tag{3.13}$$

where $(3.13)_{2,3}$ follow from (3.9). From (3.13) and (3.7) we see that

$$\begin{aligned} \boldsymbol{N}_A\,[\boldsymbol{H}] &= \boldsymbol{M}_A\,[\boldsymbol{AH} - \boldsymbol{HA}] = \boldsymbol{A}\;\boldsymbol{M}_A\,[\boldsymbol{H}] - \boldsymbol{M}_A\,[\boldsymbol{H}]\;\boldsymbol{A} \\ &= \boldsymbol{L}_A\,[\boldsymbol{A}^2\boldsymbol{H} + \boldsymbol{HA}^2 - 2\,\boldsymbol{AHA}]\,. \end{aligned} \tag{3.14}$$

By setting $\boldsymbol{G} = \boldsymbol{A}^2\boldsymbol{H} + \boldsymbol{HA}^2 - 2\,\boldsymbol{AHA}$ in (3.1) and using $(3.14)_3$, it follows that

$$\boldsymbol{X} = \boldsymbol{N}_A\,[\boldsymbol{H}] \Leftrightarrow \boldsymbol{AX} + \boldsymbol{XA} = \boldsymbol{A}^2\boldsymbol{H} + \boldsymbol{HA}^2 - 2\,\boldsymbol{AHA}\,. \tag{3.15}$$

The motivation for introducing the tensor $\boldsymbol{N}_A$ is based not so much on the result (3.15) as on the frequent occurrence of the expressions in $(3.14)_{1,2}$.

It follows from the results above that the set of fourth-order tensors

$$\mathcal{S}_A := \{\boldsymbol{B}_A,\, \boldsymbol{C}_A,\, \boldsymbol{L}_A,\, \boldsymbol{M}_A,\, \boldsymbol{N}_A\} \tag{3.16}$$

is commutative. Each tensor $\boldsymbol{K}_A \in \mathcal{S}_A$ also has the following properties:

$$(\boldsymbol{K}_A)^T = \boldsymbol{K}_A\,; \tag{3.17}$$

$$\boldsymbol{K}_A\,[\boldsymbol{BH}] = \boldsymbol{B}\;\boldsymbol{K}_A\,[\boldsymbol{H}]\,, \text{ if } \boldsymbol{AB} = \boldsymbol{BA}\,; \tag{3.18}$$

$$\boldsymbol{K}_A\,[\boldsymbol{HB}] = \boldsymbol{K}_A\,[\boldsymbol{H}]\;\boldsymbol{B}\,, \text{ if } \boldsymbol{AB} = \boldsymbol{BA}\,; \tag{3.19}$$

$$\boldsymbol{K}_{\boldsymbol{QAQ}^T}\,[\boldsymbol{QHQ}^T] = \boldsymbol{Q}\;\boldsymbol{K}_A\,[\boldsymbol{H}]\;\boldsymbol{Q}^T\,,\ \forall\ \boldsymbol{Q} \in \text{Orth}\,, \tag{3.20}$$

where Orth denotes the set of orthogonal tensors in Lin. In other words, $\boldsymbol{K}_A\,[\boldsymbol{H}]$ is an isotropic function of $\boldsymbol{A}$ and $\boldsymbol{H}$ which is linear in $\boldsymbol{H}$. Furthermore, $\boldsymbol{B}_A$, $\boldsymbol{L}_A$ and $\boldsymbol{N}_A$ are even, whereas $\boldsymbol{C}_A$ and $\boldsymbol{M}_A$ are odd. The easiest way to establish these properties is to first prove them for the tensors $\boldsymbol{B}_A$ and $\boldsymbol{C}_A$, and then use the definitions (3.4), (3.9), and (3.13) to prove the properties for $\boldsymbol{L}_A$, $\boldsymbol{M}_A$, and $\boldsymbol{N}_A$. For example, it is easily established that (3.18) holds for $\boldsymbol{K}_A = \boldsymbol{B}_A$ or $\boldsymbol{C}_A$. To prove (3.18) for $\boldsymbol{K}_A = \boldsymbol{L}_A$, use the result $\boldsymbol{B}_A\,[\boldsymbol{BH}] = \boldsymbol{B}\;\boldsymbol{B}_A\,[\boldsymbol{H}]$, apply $\boldsymbol{L}_A$ to obtain $\boldsymbol{BH} = \boldsymbol{L}_A\,[\boldsymbol{B}\;\boldsymbol{B}_A\,[\boldsymbol{H}]]$, and then set $\boldsymbol{H} = \boldsymbol{L}_A\,[\boldsymbol{X}]$ to obtain $\boldsymbol{B}\;\boldsymbol{L}_A\,[\boldsymbol{X}] = \boldsymbol{L}_A\,[\boldsymbol{BX}]$. To prove (3.18) for $\boldsymbol{K}_A = \boldsymbol{M}_A$, use the results for $\boldsymbol{K}_A = \boldsymbol{C}_A$ and $\boldsymbol{L}_A$:

$$\begin{aligned} \boldsymbol{M}_A\,[\boldsymbol{BH}] &= \boldsymbol{L}_A\,[\boldsymbol{C}_A\,[\boldsymbol{BH}]] = \boldsymbol{L}_A\,[\boldsymbol{B}\;\boldsymbol{C}_A\,[\boldsymbol{H}]] \\ &= \boldsymbol{B}\;\boldsymbol{L}_A\,[\boldsymbol{C}_A\,[\boldsymbol{H}]] = \boldsymbol{B}\;\boldsymbol{M}_A\,[\boldsymbol{H}]\,. \end{aligned}$$

The proof of (3.18) for $K_A = N_A$ is similar.

In addition to the expressions (3.10) for M_A in terms of L_A, we also have

$$M_A [H] = H - 2 L_A [H] A = 2 A L_A [H] - H . \tag{3.21}$$

To prove $(3.21)_1$, for example, use (3.10), (3.6), and (3.19) (with $K_A = L_A$ and $B = A$):

$$\begin{aligned} M_A [H] &= L_A [AH - HA] \\ &= L_A [AH + HA] - 2 L_A [HA] \\ &= H - 2 L_A [H] A . \end{aligned}$$

In addition to the expressions $(3.14)_{1,2}$ for N_A in terms of M_A, we also have

$$\begin{aligned} N_A [H] &= AH - HA - 2 M_A [H] A \\ &= HA - AH + 2 A M_A [H]. \end{aligned} \tag{3.22}$$

To prove $(3.22)_1$, for example, use $(3.14)_1$, (3.12), and (3.19):

$$\begin{aligned} N_A [H] &= M_A [AH - HA] \\ &= M_A [AH + HA] - 2 M_A [HA] \\ &= AH - HA - 2 M_A [H] A . \end{aligned}$$

From (3.21) we obtain the following expressions for L_A in terms of M_A:

$$\begin{aligned} L_A [H] &= \frac{1}{2} (H - M_A [H]) A^{-1} \\ &= \frac{1}{2} A^{-1} (H + M_A [H]) \\ &= \frac{1}{4} (A^{-1}H + HA^{-1} + A^{-1} M_A [H] - M_A [H] A^{-1}) . \end{aligned} \tag{3.23}$$

Now from (3.14) it follows that

$$A^{-1}\, N_A\,[H]\, A^{-1} = M_A\,[H]\, A^{-1} - A^{-1}\, M_A\,[H]\,. \tag{3.24}$$

Then by $(3.23)_3$ and (3.24) we obtain the following expressions for L_A in terms of N_A:

$$\begin{aligned} L_A\,[H] &= \frac{1}{4}\,(A^{-1}H + HA^{-1} - A^{-1}\, N_A\,[H]\, A^{-1}) \\ &= \frac{1}{4}\, A^{-1}\,(AH + HA - N_A\,[H])\, A^{-1}\,. \end{aligned} \tag{3.25}$$

To establish our next result we need the fact that

$$A^2H + HA^2 = 2\,AHA \Leftrightarrow AH = HA\,. \tag{3.26}$$

This can be verified by taking components relative to a principal basis for A:

$$\begin{aligned} A^2H + HA^2 = 2\,AHA &\Leftrightarrow (a_i^2 + a_j^2 - 2a_ia_j)\,\mathrm{H}_{ij} = 0 \\ &\Leftrightarrow (a_i - a_j)^2\,\mathrm{H}_{ij} = 0 \\ &\Leftrightarrow (a_i - a_j)\,\mathrm{H}_{ij} = 0 \\ &\Leftrightarrow AH - HA = 0. \end{aligned}$$

Since L_A is nonsingular, from (3.7), (3.10), $(3.12)_3$, (3.14), (3.23), (3.25), and (3.26) we conclude that *the following conditions are equivalent:*

$$\begin{aligned} AH &= HA\,, \\ A\, L_A\,[H] &= L_A\,[H]\, A\,, \\ A\, M_A\,[H] &= \pm\, M_A\,[H]\, A\,, \\ C_A\,[H] &= 0\,, \\ M_A\,[H] &= 0\,, \\ N_A\,[H] &= 0\,, \end{aligned}$$

$$L_A[H] = \frac{1}{2} A^{-1}H ,$$

$$L_A[H] = \frac{1}{2} HA^{-1} ,$$

$$L_A[H] = \frac{1}{4} (A^{-1}H + HA^{-1}) . \tag{3.27}$$

Now suppose that $H \in \mathcal{P}(A)$. Then $AH = HA$, and thus $L_A[H] = A^{-1}H$. And since $A^{-1} \in \mathcal{P}(A)$, it follows that $L_A[H] \in \mathcal{P}(A)$. Conversely, if $L_A[H] \in \mathcal{P}(A)$ then by $(3.6)_2$ it follows that $H \in \mathcal{P}(A)$. Thus we have shown that

$$L_A[H] \in \mathcal{P}(A) \Leftrightarrow H \in \mathcal{P}(A) . \tag{3.28}$$

In other words, if $AX + XA = H$, then X is a polynomial in A iff H is a polynomial in A. From $(3.6)_2$ we also see that

$$2 \text{ tr } (A^{k+1} L_A[H]) = \text{tr } (A^k H), \quad \forall \text{ integer } k . \tag{3.29}$$

Then from (3.29) and (2.8) we conclude that

$$L_A[H] \in \mathcal{P}(A)^{\perp} \Leftrightarrow H \in \mathcal{P}(A)^{\perp} . \tag{3.30}$$

In other words, if $AX + XA = H$, then X is orthogonal to every polynomial in A iff H is orthogonal to every polynomial in A. Since tr $(A^k (AH - HA)) = 0$ for any integer k, from (2.8) with $H \rightarrow AH - HA$ we conclude that

$$C_A[H] = AH - HA \in \mathcal{P}(A)^{\perp} . \tag{3.31}$$

Then from $(3.10)_1$, $(3.14)_1$, (3.30), and (3.31) we see that

$$M_A[H], N_A[H] \in \mathcal{P}(A)^{\perp} . \tag{3.22}$$

In particular, for any tensor H the solution X of (1.6) is orthogonal to every polynomial in A. Of course, since A and H commute for any $H \in \mathcal{P}(A)$, from (3.27) we also have

$$C_A[H] = M_A[H] = N_A[H] = 0, \quad \forall H \in \mathcal{P}(A). \tag{3.33}$$

We conclude this section with a description of our method for generating direct formulas for $L_A[H]$ and $M_A[H]$. Suppose that by some means we have obtained a direct formula for $L_A[G]$ which is valid for any $G \in \mathcal{P}(A)^{\perp}$ but not necessarily for other G. As we will see in Sections 5 and 6, such formulas are easily derived in the two-dimensional and three-dimensional cases. Since $AH - HA \in \mathcal{P}(A)^{\perp}$ for any tensor H, and since $M_A[H] = L_A[AH - HA]$, by setting $G = AH - HA$ in our formula for $L_A[G]$ we obtain a formula for $M_A[H]$ which is valid for any tensor H. Then from (3.23) we obtain formulas for $L_A[H]$ which are valid for any H. Alternatively, we can use the relation $N_A[H] = M_A[AH - HA]$ to obtain a formula for $N_A[H]$, and then use (3.25) to obtain formulas for $L_A[H]$ which are valid for any H. In these formulas for $L_A[H]$ we can, of course, replace the A^{-1} terms by a polynomial in A via the Cayley-Hamilton theorem.

4. SOME USEFUL IDENTITIES

Consider the tensor equation

$$AX + XA = G(A, H), \tag{4.1}$$

where $G(A, H)$ denotes one of the expressions listed in (1.5). In terms of the notation introduced in the previous section, the solution of (4.1) is $X = L_A[G(A, H)]$; and for the special case $G(A, H) = AH - HA$ this solution can also be written as $X = M_A[H]$. As we will see in the next two sections, the direct formulas for $M_A[H]$ (for arbitrary H) are much simpler than the direct formulas for $L_A[G]$ (for arbitrary G). This should not be too surprising in view of the method described in the previous section for generating these formulas. For the cases where $G(A, H)$ is given by one of the other expressions in (1.5), the existence of relatively simple direct formulas for X is due to the fact that there are simple expressions for $L_A[G(A, H)]$ in terms of M_A (or N_A), A and H. The purpose of this section is to derive these identities.

We begin with the case $G(A, H) = AHA$, that is, with the tensor equation

$$AX + XA = AHA \ , \tag{4.2}$$

which is equivalent to the tensor equation

$$A^{-1}X + XA^{-1} = H \ . \tag{4.3}$$

These equations arise in the problem of finding formulas for the material time derivative of the right stretch tensor in terms of the stretching tensor (Mehrabadi and Nemat-Nasser 1987; Hoger 1986). Since the solution of (4.2) is $X = \boldsymbol{L}_A\,[AHA]$, whereas the solution of the equivalent equation (4.3) is $X = \boldsymbol{L}_{A^{-1}}\,[H]$, we see that

$$\boldsymbol{L}_{A^{-1}}\,[H] = \boldsymbol{L}_A\,[AHA] = A\ \boldsymbol{L}_A\,[H]\ A \ , \tag{4.4}$$

where $(4.4)_2$ follows from (3.18) and (3.19) (with $B = A$). By substituting the expressions for $\boldsymbol{L}_A\,[H]$ in (3.23) and (3.25) into (4.4), we obtain the identities

$$\begin{aligned} \boldsymbol{L}_{A^{-1}}\,[H] &= \frac{1}{2}\,A\,(H - \boldsymbol{M}_A\,[H]) \\ &= \frac{1}{2}\,(H + \boldsymbol{M}_A\,[H])\,A \\ &= \frac{1}{4}\,(AH + HA - \boldsymbol{N}_A\,[H]) \ . \end{aligned} \tag{4.5}$$

In other words, the solution X of (4.2) and (4.3) is also given by any of the three expressions on the right-hand side of (4.5). The second and third of these identities are equivalent to ones obtained by Mehrabadi and Nemat-Nasser (1987) under the assumption that H is symmetric. For the three-dimensional case, they used these identities and their direct formula for $\boldsymbol{M}_A\,[H]$ to obtain direct solutions of (4.3).*

*Their equations (8.8), (8.12)–(8.13) and (8.16) correspond to equations (4.3), $(4.5)_2$ and $(4.5)_3$ in this paper. In their derivation of (8.13), and apparently of (8.16) as well, they utilized the symmetry of H ($= 2\,\hat{D}$ in their notation). However, these results, as well as their direct formulas for $\boldsymbol{L}_{A^{-1}}\,[H]$, are valid for arbitrary H; see also the comments following equation (5.26) in this paper.

Compared with (4.4), the formula for $\boldsymbol{M}_{A^{-1}}$ in terms of $\boldsymbol{M}_A$ is much simpler:

$$\boldsymbol{M}_{A^{-1}} = -\boldsymbol{M}_A \,. \tag{4.6}$$

Indeed, from (3.2) we see that $X = \boldsymbol{M}_A\,[H]$ iff $A^{-1}X + XA^{-1} = A^{-1}(-H) - (-H)A^{-1}$ iff $X = \boldsymbol{M}_{A^{-1}}[-H]$, which yields (4.6). Note that the identities (4.5) can also be obtained by replacing A with A^{-1} in (3.23) and then using (4.6) and (3.14).

Now consider the case $G(A, H) = A^2H + HA^2$, that is, the tensor equation

$$AX + XA = A^2H + HA^2 \,. \tag{4.7}$$

This equation arises in the problem of finding formulas for the Jaumann rate of the left stretch tensor in terms of the stretching tensor (Scheidler, to be published). Alternate expressions for the solution $X = \boldsymbol{L}_A\,[A^2H + HA^2]$ of (4.7) follow from the identities

$$\begin{aligned} \boldsymbol{L}_A\,[A^2H + HA^2] &= A^2\,\boldsymbol{L}_A\,[H] + \boldsymbol{L}_A\,[H]\,A^2 \\ &= 2\,\boldsymbol{L}_A\,[H]\,A^2 + (AH - HA) \\ &= 2\,A^2\,\boldsymbol{L}_A\,[H] - (AH - HA) \\ &= \frac{1}{2}\,(AH + HA + \boldsymbol{N}_A\,[H]) \,. \end{aligned} \tag{4.8}$$

$(4.8)_1$ follows from (3.18) and (3.19); $(4.8)_{2,3}$ follow from $(4.8)_1$ and (3.12); $(4.8)_4$ follows from $(3.14)_3$, $(4.4)_1$ and $(4.5)_3$.

Next, we derive alternate expressions for $\boldsymbol{L}_A\,[G(A, H)]$ when $G(A, H) = HA - AH^T$, $AH - H^TA$, $AH + H^TA$ or $HA + AH^T$; for applications of the tensor equation (4.1) in these cases see Guo (1984) and Scheidler (to be published). We claim that

$$\begin{aligned} \boldsymbol{L}_A\,[HA - AH^T] &= 2\,\boldsymbol{L}_A\,[\mathrm{skw}\,(HA)] \\ &= \mathrm{skw}\,H - \boldsymbol{M}_A\,[\mathrm{sym}\,H] \\ &= \mathrm{skw}\,(H - \boldsymbol{M}_A\,[H]) \,, \end{aligned} \tag{4.9}$$

$$\begin{aligned} \boldsymbol{L}_{\boldsymbol{A}}\,[\boldsymbol{AH} - \boldsymbol{H}^T\boldsymbol{A}] &= 2\,\boldsymbol{L}_{\boldsymbol{A}}\,[\operatorname{skw}(\boldsymbol{AH})] \\ &= \operatorname{skw}\boldsymbol{H} + \boldsymbol{M}_{\boldsymbol{A}}\,[\operatorname{sym}\boldsymbol{H}] \\ &= \operatorname{skw}(\boldsymbol{H} + \boldsymbol{M}_{\boldsymbol{A}}\,[\boldsymbol{H}])\,, \end{aligned} \tag{4.10}$$

$$\begin{aligned} \boldsymbol{L}_{\boldsymbol{A}}\,[\boldsymbol{AH} + \boldsymbol{H}^T\boldsymbol{A}] &= 2\,\boldsymbol{L}_{\boldsymbol{A}}\,[\operatorname{sym}(\boldsymbol{AH})] \\ &= \boldsymbol{H} - 2\,\boldsymbol{L}_{\boldsymbol{A}}\,[\operatorname{skw}\boldsymbol{H}]\,\boldsymbol{A} \\ &= \operatorname{sym}\boldsymbol{H} + \boldsymbol{M}_{\boldsymbol{A}}\,[\operatorname{skw}\boldsymbol{H}] \\ &= \operatorname{sym}(\boldsymbol{H} + \boldsymbol{M}_{\boldsymbol{A}}\,[\boldsymbol{H}])\,, \end{aligned} \tag{4.11}$$

$$\begin{aligned} \boldsymbol{L}_{\boldsymbol{A}}\,[\boldsymbol{HA} + \boldsymbol{A}\boldsymbol{H}^T] &= 2\,\boldsymbol{L}_{\boldsymbol{A}}\,[\operatorname{sym}(\boldsymbol{HA})] \\ &= \boldsymbol{H} - 2\,\boldsymbol{A}\,\boldsymbol{L}_{\boldsymbol{A}}\,[\operatorname{skw}\boldsymbol{H}] \\ &= \operatorname{sym}\boldsymbol{H} - \boldsymbol{M}_{\boldsymbol{A}}\,[\operatorname{skw}\boldsymbol{H}] \\ &= \operatorname{sym}(\boldsymbol{H} - \boldsymbol{M}_{\boldsymbol{A}}\,[\boldsymbol{H}])\,. \end{aligned} \tag{4.12}$$

These identities follow easily from (2.3)–(2.6), (3.6), (3.10), (3.18), (3.19) and the fact that $\boldsymbol{M}_{\boldsymbol{A}}$ is odd (see (2.10)).

We conclude this section with two identities which are used in Scheidler (to be published):

$$\begin{aligned} \boldsymbol{M}_{\boldsymbol{A}}\,[\boldsymbol{AH} - \boldsymbol{H}^T\boldsymbol{A}] &= 2\,\boldsymbol{M}_{\boldsymbol{A}}\,[\operatorname{skw}(\boldsymbol{AH})] \\ &= \boldsymbol{A}\operatorname{skw}\boldsymbol{H} - (\operatorname{skw}\boldsymbol{H})\,\boldsymbol{A} + \boldsymbol{N}_{\boldsymbol{A}}\,[\operatorname{sym}\boldsymbol{H}]\,, \end{aligned} \tag{4.13}$$

$$\begin{aligned} \boldsymbol{M}_{\boldsymbol{A}}\,[\boldsymbol{HA} - \boldsymbol{A}\boldsymbol{H}^T] &= 2\,\boldsymbol{M}_{\boldsymbol{A}}\,[\operatorname{skw}(\boldsymbol{HA})] \\ &= \boldsymbol{A}\operatorname{skw}\boldsymbol{H} - (\operatorname{skw}\boldsymbol{H})\,\boldsymbol{A} - \boldsymbol{N}_{\boldsymbol{A}}\,[\operatorname{sym}\boldsymbol{H}]\,. \end{aligned} \tag{4.14}$$

These identities follow easily from (2.5), (2.6), (3.12), and (3.14).

5. FORMULAS FOR THE THREE-DIMENSIONAL CASE

In this section we derive direct formulas for the fourth-order tensors $\boldsymbol{L}_{\boldsymbol{A}}$, $\boldsymbol{M}_{\boldsymbol{A}}$, and $\boldsymbol{N}_{\boldsymbol{A}}$ under the assumption that the underlying vector space $\mathcal{V}$ is three-dimensional. We begin by listing some identities for second-order tensors which will be used in the derivation of these formulas. The principal invariants of a second-order tensor $\boldsymbol{B}$ are denoted by $\mathrm{I}_{\boldsymbol{B}}$, $\mathrm{II}_{\boldsymbol{B}}$ and $\mathrm{III}_{\boldsymbol{B}}$:

$$I_B := \text{tr } \boldsymbol{B}, \quad II_B := \frac{1}{2}\,[(\text{tr } \boldsymbol{B})^2 - \text{tr } (\boldsymbol{B}^2)], \quad III_B := \det \boldsymbol{B} \,. \tag{5.1}$$

From the Cayley-Hamilton theorem we obtain

$$\boldsymbol{A}^3 = I_A\, \boldsymbol{A}^2 - II_A\, \boldsymbol{A} + III_A\, \boldsymbol{I} \,, \tag{5.2}$$

and

$$III_A\, \boldsymbol{A}^{-1} = \boldsymbol{A}^2 - I_A\, \boldsymbol{A} + II_A\, \boldsymbol{I} \,. \tag{5.3}$$

Since the expression $I_A\, \boldsymbol{I} - \boldsymbol{A}$ occurs frequently below, it is convenient to introduce a special symbol for this expression:

$$\tilde{\boldsymbol{A}} := I_A\, \boldsymbol{I} - \boldsymbol{A} \,. \tag{5.4}$$

The eigenvalues of $\tilde{\boldsymbol{A}}$ are $a_2 + a_3$, $a_3 + a_1$, $a_1 + a_2$; in particular, $\tilde{\boldsymbol{A}}$ is symmetric positive-definite. Also note that (Sidoroff 1978; Guo 1984; Stickforth and Wegener 1988)

$$III_{\tilde{A}} = I_A\, II_A - III_A = (a_1 + a_2)\,(a_2 + a_3)\,(a_3 + a_1) \,, \tag{5.5}$$

and that (Guo 1984)

$$III_{\tilde{A}}\, \tilde{\boldsymbol{A}}^{-1} = \boldsymbol{A}^2 + II_A\, \boldsymbol{I} \,. \tag{5.6}$$

The following identities are due to Rivlin (1955):

$$\begin{aligned} \boldsymbol{A}^2\boldsymbol{H}\boldsymbol{A}^2 = {} & II_A\, \boldsymbol{A}\boldsymbol{H}\boldsymbol{A} - III_A\, (\boldsymbol{A}\boldsymbol{H} + \boldsymbol{H}\boldsymbol{A}) \\ & + I_{\boldsymbol{A}^2\boldsymbol{H}}\, \boldsymbol{A}^2 + \alpha_1(\boldsymbol{A}, \boldsymbol{H})\, \boldsymbol{A} + III_A\, I_{\boldsymbol{A}\boldsymbol{H}}\, \boldsymbol{I} \,, \end{aligned} \tag{5.7}$$

$$\begin{aligned} \boldsymbol{A}^2\boldsymbol{H}\boldsymbol{A} + \boldsymbol{A}\boldsymbol{H}\boldsymbol{A}^2 = {} & I_A\, \boldsymbol{A}\boldsymbol{H}\boldsymbol{A} - III_A\, \boldsymbol{H} \\ & + I_{\boldsymbol{A}\boldsymbol{H}}\, \boldsymbol{A}^2 + \alpha_2(\boldsymbol{A}, \boldsymbol{H})\, \boldsymbol{A} + III_A\, I_H\, \boldsymbol{I} \,, \end{aligned} \tag{5.8}$$

$$\begin{aligned} \boldsymbol{A}^2\boldsymbol{H} + \boldsymbol{H}\boldsymbol{A}^2 = {} & - \boldsymbol{A}\boldsymbol{H}\boldsymbol{A} + I_A\, (\boldsymbol{A}\boldsymbol{H} + \boldsymbol{H}\boldsymbol{A}) - II_A\, \boldsymbol{H} \\ & + I_H\, \boldsymbol{A}^2 + \alpha_3(\boldsymbol{A}, \boldsymbol{H})\, \boldsymbol{A} + \alpha_4(\boldsymbol{A}, \boldsymbol{H})\, \boldsymbol{I} \,, \end{aligned} \tag{5.9}$$

where

$$\alpha_1(\boldsymbol{A}, \boldsymbol{H}) = \mathrm{tr}\,[(\mathrm{III}_{\boldsymbol{A}}\,\boldsymbol{I} - \mathrm{II}_{\boldsymbol{A}}\,\boldsymbol{A})\,\boldsymbol{H}]$$
$$= \mathrm{III}_{\boldsymbol{A}}\,\mathrm{I}_{\boldsymbol{H}} - \mathrm{II}_{\boldsymbol{A}}\,\mathrm{I}_{\boldsymbol{AH}} = -\,\mathrm{I}_{\tilde{\boldsymbol{A}}\boldsymbol{A}^2\boldsymbol{H}} \tag{5.10}$$

$$\alpha_2(\boldsymbol{A}, \boldsymbol{H}) = \mathrm{tr}\,[(\boldsymbol{A}^2 - \mathrm{I}_{\boldsymbol{A}}\,\boldsymbol{A})\,\boldsymbol{H}]$$
$$= \mathrm{I}_{\boldsymbol{A}^2\boldsymbol{H}} - \mathrm{I}_{\boldsymbol{A}}\,\mathrm{I}_{\boldsymbol{AH}} = -\,\mathrm{I}_{\tilde{\boldsymbol{A}}\boldsymbol{AH}}\,, \tag{5.11}$$

$$\alpha_3(\boldsymbol{A}, \boldsymbol{H}) = -\,\mathrm{tr}\,[(\mathrm{I}_{\boldsymbol{A}}\,\boldsymbol{I} - \boldsymbol{A})\,\boldsymbol{H}]$$
$$= \mathrm{I}_{\boldsymbol{AH}} - \mathrm{I}_{\boldsymbol{A}}\,\mathrm{I}_{\boldsymbol{H}} = -\,\mathrm{I}_{\tilde{\boldsymbol{A}}\boldsymbol{H}}\,, \tag{5.12}$$

$$\alpha_4(\boldsymbol{A}, \boldsymbol{H}) = \mathrm{tr}\,[(\boldsymbol{A}^2 - \mathrm{I}_{\boldsymbol{A}}\,\boldsymbol{A} + \mathrm{II}_{\boldsymbol{A}}\,\boldsymbol{I})\,\boldsymbol{H}]$$
$$= \mathrm{I}_{\boldsymbol{A}^2\boldsymbol{H}} - \mathrm{I}_{\boldsymbol{A}}\,\mathrm{I}_{\boldsymbol{AH}} + \mathrm{II}_{\boldsymbol{A}}\,\mathrm{I}_{\boldsymbol{H}} = \mathrm{III}_{\boldsymbol{A}}\,\mathrm{I}_{\boldsymbol{A}^{-1}\boldsymbol{H}}\,. \tag{5.13}$$

Rivlin's identities, as well as (5.2)–(5.5)$_1$, are valid for any second-order tensors $\boldsymbol{A}$ and $\boldsymbol{H}$. However, we will continue to assume that $\boldsymbol{A}$ is symmetric positive-definite.

Observe that if $\boldsymbol{H} \in \mathcal{P}(\boldsymbol{A})^{\perp}$ (see (2.8)) then the second line on the right-hand side of Rivlin's identities (5.7)–(5.9) vanishes. In this case we also have the identity

$$\tilde{\boldsymbol{A}}\,(\boldsymbol{AH} + \boldsymbol{HA})\,\tilde{\boldsymbol{A}} = \mathrm{III}_{\tilde{\boldsymbol{A}}}\,\boldsymbol{H}, \quad \forall\, \boldsymbol{H} \in \mathcal{P}(\boldsymbol{A})^{\perp}\,. \tag{5.14}$$

This follows by substituting (5.4) into the left-hand side of (5.14), expanding, and then using (5.5)$_1$, (5.8), and (5.9). Similarly, from (5.5)$_1$, (5.7), and (5.9) we obtain the identity

$$(\boldsymbol{A}^2 + \mathrm{II}_{\boldsymbol{A}}\,\boldsymbol{I})\,\boldsymbol{H}\,(\boldsymbol{A}^2 + \mathrm{II}_{\boldsymbol{A}}\,\boldsymbol{I}) = \mathrm{III}_{\tilde{\boldsymbol{A}}}\,(\boldsymbol{AH} + \boldsymbol{HA}), \quad \forall\, \mathbf{H} \in \mathcal{P}(\mathbf{A})^{\perp}\,. \tag{5.15}$$

In view of (5.6) we see that the identities (5.14) and (5.15) are equivalent.

Now suppose that $\boldsymbol{G} \in \mathcal{P}(\boldsymbol{A})^{\perp}$, in which case $\boldsymbol{L}_{\boldsymbol{A}}\,[\boldsymbol{G}] \in \mathcal{P}(\boldsymbol{A})^{\perp}$ also (see (3.30)). If we set $\boldsymbol{H} = \boldsymbol{L}_{\boldsymbol{A}}\,[\boldsymbol{G}]$ in (5.14) we obtain

$$\tilde{\boldsymbol{A}}\,(\boldsymbol{A}\,\boldsymbol{L}_{\boldsymbol{A}}\,[\boldsymbol{G}] + \boldsymbol{L}_{\boldsymbol{A}}\,[\boldsymbol{G}]\,\boldsymbol{A})\,\tilde{\boldsymbol{A}} = \mathrm{III}_{\tilde{\boldsymbol{A}}}\,\boldsymbol{L}_{\boldsymbol{A}}\,[\boldsymbol{G}]\,.$$

But since $\boldsymbol{A}\ \boldsymbol{L}_{\boldsymbol{A}}[\boldsymbol{G}] + \boldsymbol{L}_{\boldsymbol{A}}[\boldsymbol{G}]\ \boldsymbol{A} = \boldsymbol{G}$ for any tensor $\boldsymbol{G}$ (s (3.6)), we have the following simple formula for $\boldsymbol{L}_{\boldsymbol{A}}[\boldsymbol{G}]$:

$$\mathrm{III}_{\tilde{A}}\ \boldsymbol{L}_{\boldsymbol{A}}[\boldsymbol{G}] = \tilde{\boldsymbol{A}}\boldsymbol{G}\tilde{\boldsymbol{A}}, \quad \forall\ \boldsymbol{G} \in \mathcal{P}(\boldsymbol{A})^{\perp}\ . \tag{5.16}$$

By substituting the definition (5.4) of $\tilde{\boldsymbol{A}}$ into (5.16) and expanding, we obtain the alternate formula

$$\mathrm{III}_{\tilde{A}}\ \boldsymbol{L}_{\boldsymbol{A}}[\boldsymbol{G}] = \mathrm{I}_{\boldsymbol{A}}^{2}\,\boldsymbol{G} - \mathrm{I}_{\boldsymbol{A}}\,(\boldsymbol{A}\boldsymbol{G} + \boldsymbol{G}\boldsymbol{A}) + \boldsymbol{A}\boldsymbol{G}\boldsymbol{A}, \quad \forall\ \boldsymbol{G} \in \mathcal{P}(\boldsymbol{A})^{\perp}\ . \tag{5.17}$$

By using Rivlin's identity (5.9) with $\boldsymbol{H} = \boldsymbol{G}$, we see that (5.17) is equivalent to the formula

$$\mathrm{III}_{\tilde{A}}\ \boldsymbol{L}_{\boldsymbol{A}}[\boldsymbol{G}] = (\mathrm{I}_{\boldsymbol{A}}^{2} - \mathrm{II}_{\boldsymbol{A}})\,\boldsymbol{G} - (\boldsymbol{A}^{2}\boldsymbol{G} + \boldsymbol{G}\boldsymbol{A}^{2}), \quad \forall\ \boldsymbol{G} \in \mathcal{P}(\boldsymbol{A})^{\perp}\ . \tag{5.18}$$

In view of (3.1), the formulas (5.16)–(5.18) yield simple direct solutions of the tensor equation (1.1) when $\boldsymbol{G} \in \mathcal{P}(\boldsymbol{A})^{\perp}$. Since any skew tensor belongs to $\mathcal{P}(\boldsymbol{A})^{\perp}$, the formulas (5.16)–(5.18) hold in particular for any $\boldsymbol{G} \in$ Skw. For this special case the formula (5.18) was obtained by Sidoroff (1978) and Guo (1984). Their derivations employ the axial vector associated with a skew tensor and thus do not carry over to the more general case where $\boldsymbol{G}$ need only belong to $\mathcal{P}(\boldsymbol{A})^{\perp}$.

Since $\boldsymbol{M}_{\boldsymbol{A}}[\boldsymbol{H}] = \boldsymbol{L}_{\boldsymbol{A}}[\boldsymbol{A}\boldsymbol{H} - \boldsymbol{H}\boldsymbol{A}]$, and since $\boldsymbol{A}\boldsymbol{H} - \boldsymbol{H}\boldsymbol{A} \in \mathcal{P}(\boldsymbol{A})^{\perp}$ for any tensor $\boldsymbol{H}$ (see (3.31)), by setting $\boldsymbol{G} = \boldsymbol{A}\boldsymbol{H} - \boldsymbol{H}\boldsymbol{A}$ in (5.16) and (5.17) we obtain the following simple formulas for $\boldsymbol{M}_{\boldsymbol{A}}[\boldsymbol{H}]$:

$$\begin{aligned}\mathrm{III}_{\tilde{A}}\ \boldsymbol{M}_{\boldsymbol{A}}[\boldsymbol{H}] &= \tilde{\boldsymbol{A}}\,(\boldsymbol{A}\boldsymbol{H} - \boldsymbol{H}\boldsymbol{A})\,\tilde{\boldsymbol{A}} \\ &= \mathrm{I}_{\boldsymbol{A}}^{2}\,(\boldsymbol{A}\boldsymbol{H} - \boldsymbol{H}\boldsymbol{A}) - \mathrm{I}_{\boldsymbol{A}}\,(\boldsymbol{A}^{2}\boldsymbol{H} - \boldsymbol{H}\boldsymbol{A}^{2}) + \boldsymbol{A}^{2}\boldsymbol{H}\boldsymbol{A} - \boldsymbol{A}\boldsymbol{H}\boldsymbol{A}^{2}\ .\end{aligned} \tag{5.19}$$

Furthermore, since

$$\boldsymbol{A}\boldsymbol{H} - \boldsymbol{H}\boldsymbol{A} = \boldsymbol{H}\tilde{\boldsymbol{A}} - \tilde{\boldsymbol{A}}\boldsymbol{H}\ , \tag{5.20}$$

by $(5.19)_1$ we also have

$$\begin{aligned}\mathrm{III}_{\tilde{A}}\ \boldsymbol{M}_{\boldsymbol{A}}[\boldsymbol{H}] &= \tilde{\boldsymbol{A}}\,(\boldsymbol{H}\tilde{\boldsymbol{A}} - \tilde{\boldsymbol{A}}\boldsymbol{H})\,\tilde{\boldsymbol{A}} \\ &= \tilde{\boldsymbol{A}}\boldsymbol{H}\tilde{\boldsymbol{A}}^{2} - \tilde{\boldsymbol{A}}^{2}\boldsymbol{H}\tilde{\boldsymbol{A}}\ .\end{aligned} \tag{5.21}$$

The formula $(5.19)_2$ was obtained by Mehrabadi and Nemat-Nasser (1987) by repeated applications of the Cayley-Hamilton theorem. We emphasize that (5.19) and (5.21) are valid for any tensor $\boldsymbol{H}$. In view of (3.2), these formulas yield simple direct solutions of the tensor equation (1.6). Furthermore, in view of (3.1) and the identities $(4.5)_{1,2}$, (4.9), (4.10), $(4.11)_{2\text{-}4}$, and $(4.12)_{2\text{-}4}$, the formulas (5.16)–(5.18), (5.19) and (5.21) yield simple direct solutions of (4.1) when $\boldsymbol{G}(\boldsymbol{A}, \boldsymbol{H}) = \boldsymbol{AHA}$, $\boldsymbol{HA} - \boldsymbol{AH}^{\mathrm{T}}$, $\boldsymbol{AH} - \boldsymbol{H}^{\mathrm{T}}\boldsymbol{A}$, $\boldsymbol{AH} + \boldsymbol{H}^{\mathrm{T}}\boldsymbol{A}$ or $\boldsymbol{HA} + \boldsymbol{AH}^{\mathrm{T}}$.

Since $\boldsymbol{N}_{\boldsymbol{A}}[\boldsymbol{H}] = \boldsymbol{M}_{\boldsymbol{A}}[\boldsymbol{AH} - \boldsymbol{HA}]$, by replacing $\boldsymbol{H}$ with $\boldsymbol{AH} - \boldsymbol{HA}$ in $(5.19)_1$ we obtain the formula

$$\mathrm{III}_{\tilde{\boldsymbol{A}}}\, \boldsymbol{N}_{\boldsymbol{A}}[\boldsymbol{H}] = \tilde{\boldsymbol{A}}\,(\boldsymbol{A}^2\boldsymbol{H} + \boldsymbol{HA}^2 - 2\,\boldsymbol{AHA})\,\tilde{\boldsymbol{A}}\,. \tag{5.22}$$

By substituting (5.4) into (5.22), expanding, and then using (5.2) to reduce the $\boldsymbol{A}^3$ terms, we obtain

$$\begin{aligned}\mathrm{III}_{\tilde{\boldsymbol{A}}}\, \boldsymbol{N}_{\boldsymbol{A}}[\boldsymbol{H}] = &- 2\boldsymbol{A}^2\boldsymbol{HA}^2 + 2\mathrm{I}_{\boldsymbol{A}}\,(\boldsymbol{A}^2\boldsymbol{HA} + \boldsymbol{AHA}^2) - 2\,(\mathrm{I}_{\boldsymbol{A}}^2 + \mathrm{II}_{\boldsymbol{A}})\,\boldsymbol{AHA}\\ &+ (\mathrm{I}_{\boldsymbol{A}}\,\mathrm{II}_{\boldsymbol{A}} + \mathrm{III}_{\boldsymbol{A}})\,(\boldsymbol{AH} + \boldsymbol{HA}) - 2\mathrm{I}_{\boldsymbol{A}}\,\mathrm{III}_{\boldsymbol{A}}\,\boldsymbol{H}\,.\end{aligned} \tag{5.23}$$

Substitution of Rivlin's identities (5.7) and (5.8) into (5.23) yields

$$\begin{aligned}\mathrm{III}_{\tilde{\boldsymbol{A}}}\, \boldsymbol{N}_{\boldsymbol{A}}[\boldsymbol{H}] = &- 4\mathrm{II}_{\boldsymbol{A}}\,\boldsymbol{AHA} + (\mathrm{I}_{\boldsymbol{A}}\,\mathrm{II}_{\boldsymbol{A}} + 3\mathrm{III}_{\boldsymbol{A}})\,(\boldsymbol{AH} + \boldsymbol{HA}) - 4\mathrm{I}_{\boldsymbol{A}}\,\mathrm{III}_{\boldsymbol{A}}\,\boldsymbol{H}\\ &- 2\alpha_2(\boldsymbol{A}, \boldsymbol{H})\,\boldsymbol{A}^2 + 2\alpha_5(\boldsymbol{A}, \boldsymbol{H})\,\boldsymbol{A} - 2\mathrm{III}_{\boldsymbol{A}}\,\alpha_3(\boldsymbol{A}, \boldsymbol{H})\,\boldsymbol{I}\,,\end{aligned} \tag{5.24}$$

where α_2 and α_3 are defined by (5.11) and (5.12), and

$$\begin{aligned}\alpha_5(\boldsymbol{A}, \boldsymbol{H}) &= \mathrm{tr}\,[(\mathrm{I}_{\boldsymbol{A}}\,\boldsymbol{A}^2 + (\mathrm{II}_{\boldsymbol{A}} - \mathrm{I}_{\boldsymbol{A}}^2)\,\boldsymbol{A} - \mathrm{III}_{\boldsymbol{A}}\,\boldsymbol{I})\,\boldsymbol{H}]\\ &= \mathrm{I}_{\boldsymbol{A}}\,\mathrm{I}_{\boldsymbol{A}^2\boldsymbol{H}} + (\mathrm{II}_{\boldsymbol{A}} - \mathrm{I}_{\boldsymbol{A}}^2)\,\mathrm{I}_{\boldsymbol{AH}} - \mathrm{III}_{\boldsymbol{A}}\,\mathrm{I}_{\boldsymbol{H}}\,.\end{aligned} \tag{5.25}$$

By using $\boldsymbol{N}_{\boldsymbol{A}}[\boldsymbol{H}] = \boldsymbol{M}_{\boldsymbol{A}}[\boldsymbol{AH} - \boldsymbol{HA}]$, (5.20) and (5.21) with $\boldsymbol{H} \to \boldsymbol{H}\tilde{\boldsymbol{A}} - \tilde{\boldsymbol{A}}\boldsymbol{H}$, we also obtain

$$\begin{aligned}\mathrm{III}_{\tilde{\boldsymbol{A}}}\, \boldsymbol{N}_{\boldsymbol{A}}[\boldsymbol{H}] &= \tilde{\boldsymbol{A}}\,(\tilde{\boldsymbol{A}}^2\boldsymbol{H} + \boldsymbol{H}\tilde{\boldsymbol{A}}^2 - 2\tilde{\boldsymbol{A}}\boldsymbol{H}\tilde{\boldsymbol{A}})\,\tilde{\boldsymbol{A}}\\ &= \tilde{\boldsymbol{A}}^3\boldsymbol{H}\tilde{\boldsymbol{A}} + \tilde{\boldsymbol{A}}\boldsymbol{H}\tilde{\boldsymbol{A}}^3 - 2\tilde{\boldsymbol{A}}^2\boldsymbol{H}\tilde{\boldsymbol{A}}^2\\ &= -2\,\tilde{\boldsymbol{A}}^2\boldsymbol{H}\tilde{\boldsymbol{A}}^2 + \mathrm{I}_{\tilde{\boldsymbol{A}}}\,(\tilde{\boldsymbol{A}}^2\boldsymbol{H}\tilde{\boldsymbol{A}} + \tilde{\boldsymbol{A}}\boldsymbol{H}\tilde{\boldsymbol{A}}^2) - 2\,\mathrm{II}_{\tilde{\boldsymbol{A}}}\,\tilde{\boldsymbol{A}}\boldsymbol{H}\tilde{\boldsymbol{A}} + \mathrm{III}_{\tilde{\boldsymbol{A}}}\,(\tilde{\boldsymbol{A}}\boldsymbol{H} + \boldsymbol{H}\tilde{\boldsymbol{A}})\,,\end{aligned} \tag{5.26}$$

where the last formula follows from the Cayley-Hamilton theorem for $\tilde{A}$. In view of $(4.5)_3$ and $(4.8)_4$, the formulas (5.22)–(5.26) yield direct solutions of the tensor equations (4.2), (4.3), and (4.7). The direct solution of (4.2) or (4.3) obtained by substituting (5.23) into $(4.5)_3$ is equivalent to the first of two formulas obtained by Mehrabadi and Nemat-Nasser (1987); cf. equation (8.8) and the formula preceding (8.17) in their paper.* Their second formula is similar to the one obtained by substituting (5.24) into $(4.5)_3$; Rivlin's identity (5.9) may be used to obtain their result from ours and vice versa.

Now we derive some formulas for $\boldsymbol{L}_A\,[\boldsymbol{H}]$ which are valid for any tensor $\boldsymbol{H}$. Substitution of (5.23) into the expression $(3.25)_1$ for $\boldsymbol{L}_A$ in terms of $\boldsymbol{N}_A$ yields

$$\begin{aligned} 2\,\mathrm{III}_{\tilde{A}}\,\boldsymbol{L}_A\,[\boldsymbol{H}] = {} & \boldsymbol{AHA} - \mathrm{I}_A\,(\boldsymbol{AH} + \boldsymbol{HA}) + (\mathrm{I}_A^2 + \mathrm{II}_A)\,\boldsymbol{H} \\ & + \mathrm{I}_A\,\mathrm{III}_A\,\boldsymbol{A}^{-1}\boldsymbol{HA}^{-1} - \mathrm{III}_A\,(\boldsymbol{A}^{-1}\boldsymbol{H} + \boldsymbol{HA}^{-1})\,. \end{aligned} \tag{5.27}$$

This formula was stated without proof by Leonov (1976) and Stickforth and Wegener (1988). Leonov attributes (5.27) to L. M. Zubov; Stickforth and Wegener refer the reader to Stickforth (1982–1983) for a proof. Substitution of the formula (5.3) for $\boldsymbol{A}^{-1}$ into (5.27) yields

$$\begin{aligned} 2\,\mathrm{III}_A\,\mathrm{III}_{\tilde{A}}\,\boldsymbol{L}_A\,[\boldsymbol{H}] = {} & \mathrm{I}_A\,\boldsymbol{A}^2\boldsymbol{HA}^2 - \mathrm{I}_A^2\,(\boldsymbol{A}^2\boldsymbol{HA} + \boldsymbol{AHA}^2) \\ & + \mathrm{III}_{\tilde{A}}\,(\boldsymbol{A}^2\boldsymbol{H} + \boldsymbol{HA}^2) + (\mathrm{I}_A^3 + \mathrm{III}_A)\,\boldsymbol{AHA} \\ & - \mathrm{I}_A^2\,\mathrm{II}_A\,(\boldsymbol{AH} + \boldsymbol{HA}) + (\mathrm{I}_A^2\,\mathrm{III}_A + \mathrm{II}_A\,\mathrm{III}_{\tilde{A}})\,\boldsymbol{H}\,. \end{aligned} \tag{5.28}$$

This formula is due to Hoger and Carlson (1984); they derived it under the assumption that $\boldsymbol{H}$ is symmetric and then observed that the formula is valid for any $\boldsymbol{H}$. In view of (3.1), the formulas (5.27) and (5.28) yield direct formulas for the solution $\boldsymbol{X}$ of the tensor equation $\boldsymbol{AX} + \boldsymbol{XA} = \boldsymbol{H}$ for arbitrary $\boldsymbol{H}$.

Observe that when $\boldsymbol{G} \in \mathcal{P}(\boldsymbol{A})^{\perp}$, the simple formulas (5.16)–(5.18) for $\boldsymbol{L}_A\,[\boldsymbol{G}]$ do not follow immediately from (5.27) or (5.28). However, Hoger and Carlson (1984) noted that substitution of Rivlin's identities (5.7)–(5.9) into (5.28) yields

*The last term on the right-hand side of their first formula contains a misprint; the term should read $\mathrm{I}_U\,\mathrm{III}_U\,\hat{\boldsymbol{D}}$.

$$\mathrm{III}_{\tilde{A}}\, \boldsymbol{L}_{\boldsymbol{A}}\,[\boldsymbol{H}] = -\,(\boldsymbol{A}^2\boldsymbol{H} + \boldsymbol{H}\boldsymbol{A}^2) + (\mathrm{I}_{\boldsymbol{A}}^2 - \mathrm{II}_{\boldsymbol{A}})\,\boldsymbol{H}$$
$$+ \beta_2(\boldsymbol{A}, \boldsymbol{H})\,\boldsymbol{A}^2 + \beta_1(\boldsymbol{A}, \boldsymbol{H})\,\boldsymbol{A} + \beta_0(\boldsymbol{A}, \boldsymbol{H})\,\boldsymbol{I}\,; \qquad (5.29)$$

cf. their equation (2.7) for expressions for the isotropic scalar functions β_i. They observed that if $\boldsymbol{H} \in$ Skw then the β_i are zero, and consequently, (5.29) reduces to (5.18) for skew $\boldsymbol{G}$. Indeed, it is clear from their expressions for the $\beta_i(\boldsymbol{A}, \boldsymbol{H})$ that they reduce to zero for any $\boldsymbol{H} \in \mathcal{P}(\boldsymbol{A})^{\perp}$, and thus that (5.18) is a special case of (5.29). By using a representation for isotropic tensor functions of two symmetric tensors, together with his formula for $\boldsymbol{L}_{\boldsymbol{A}}\,[\boldsymbol{G}]$ when $\boldsymbol{G} \in$ Skw (see (5.18)), Sidoroff (1978) also obtained (5.29). His expressions for the β_i are more compact than those of Hoger and Carlson; however, Sidoroff's expressions involve $\boldsymbol{A}^{-1}$ and $\boldsymbol{A}^{-2}$ and thus may be less useful for some applications. Other direct formulas for $\boldsymbol{L}_{\boldsymbol{A}}$ in terms of $\boldsymbol{A}$ and $\boldsymbol{A}^{-1}$ follow from $(3.23)_{1,2}$ and $(5.19)_1$:

$$2\,\boldsymbol{L}_{\boldsymbol{A}}\,[\boldsymbol{H}] = [\boldsymbol{H} - \frac{1}{\mathrm{III}_{\tilde{A}}}\,\tilde{\boldsymbol{A}}\,(\boldsymbol{A}\boldsymbol{H} - \boldsymbol{H}\boldsymbol{A})\,\tilde{\boldsymbol{A}}]\,\boldsymbol{A}^{-1}$$

$$= \boldsymbol{H}\boldsymbol{A}^{-1} + \frac{1}{\mathrm{III}_{\tilde{A}}}\,\tilde{\boldsymbol{A}}\,(\boldsymbol{H} - \boldsymbol{A}\boldsymbol{H}\boldsymbol{A}^{-1})\,\tilde{\boldsymbol{A}}$$

$$= \boldsymbol{A}^{-1}\,[\,\boldsymbol{H} + \frac{1}{\mathrm{III}_{\tilde{A}}}\,\tilde{\boldsymbol{A}}\,(\boldsymbol{A}\boldsymbol{H} - \boldsymbol{H}\boldsymbol{A})\,\tilde{\boldsymbol{A}}\,]$$

$$= \boldsymbol{A}^{-1}\,\boldsymbol{H} + \frac{1}{\mathrm{III}_{\tilde{A}}}\,\tilde{\boldsymbol{A}}\,(\boldsymbol{H} - \boldsymbol{A}^{-1}\,\boldsymbol{H}\boldsymbol{A})\,\tilde{\boldsymbol{A}} \qquad (5.30)$$

6. FORMULAS FOR THE TWO-DIMENSIONAL CASE

In this section we derive direct formulas for the fourth-order tensors $\boldsymbol{L}_{\boldsymbol{A}}$, $\boldsymbol{M}_{\boldsymbol{A}}$, and $\boldsymbol{N}_{\boldsymbol{A}}$ under the assumption that the underlying vector space $\mathcal{V}$ is two-dimensional. We begin by listing some identities for second-order tensors which will be used in the derivation of these formulas. The principal invariants of a second-order tensor $\boldsymbol{B}$ are denoted by $\mathrm{I}_{\boldsymbol{B}}$ and $\mathrm{II}_{\boldsymbol{B}}$:

$$\mathrm{I}_{\boldsymbol{B}} := \mathrm{tr}\,\boldsymbol{B}\,, \qquad \mathrm{II}_{\boldsymbol{B}} := \det \boldsymbol{B}\,. \qquad (6.1)$$

From the Cayley-Hamilton theorem we obtain

$$A^2 = \mathrm{I}_A\, A - \mathrm{II}_A\, I\,, \qquad \mathrm{II}_A\, A^{-1} = \mathrm{I}_A\, I - A\,. \tag{6.2}$$

The following identities are due to Rivlin (1955):

$$AH + HA = \mathrm{I}_A\, H + \mathrm{I}_H\, A + \beta_1(A, H)\, I\,, \tag{6.3}$$

$$AHA = \mathrm{II}_A\, H + \mathrm{I}_{AH}\, A - \mathrm{II}_A\, \mathrm{I}_H\, I\,, \tag{6.4}$$

where

$$\begin{aligned}\beta_1(A, H) &= \mathrm{tr}\,[(A - \mathrm{I}_A\, I)\, H] \\ &= \mathrm{I}_{AH} - \mathrm{I}_A\, \mathrm{I}_H = -\,\mathrm{II}_A\, \mathrm{I}_{A^{-1}H}\,.\end{aligned} \tag{6.5}$$

Observe that if $H \in \mathcal{P}(A)^{\perp}$, then all but the first terms on the right-hand side of Rivlin's identities (6.3) and (6.4) vanish:

$$AH + HA = \mathrm{I}_A H\,, \quad AHA = \mathrm{II}_A H\,, \quad \forall\, H \in \mathcal{P}(A)^{\perp}\,. \tag{6.6}$$

Now suppose that $G \in \mathcal{P}(A)^{\perp}$, in which case $\boldsymbol{L}_A\,[G] \in \mathcal{P}(A)^{\perp}$ also. If we set $H = \boldsymbol{L}_A\,[G]$ in $(6.6)_1$ and use the fact that $A\boldsymbol{L}_A\,[G] + \boldsymbol{L}_A\,[G]\,A = G$ for any tensor G, we obtain the following simple formula for $\boldsymbol{L}_A\,[G]$:

$$\mathrm{I}_A\, \boldsymbol{L}_A\,[G] = G\,, \qquad \forall\, G \in \mathcal{P}(A)^{\perp}\,. \tag{6.7}$$

In particular, the formula (6.7) holds for any $G \in$ Skw. Since $\boldsymbol{M}_A\,[H] = \boldsymbol{L}_A\,[AH - HA]$, and since $AH - HA \in \mathcal{P}(A)^{\perp}$ for any tensor H, by setting $G = AH - HA$ in (6.7) we obtain the following simple formula for $\boldsymbol{M}_A\,[H]$:

$$\mathrm{I}_A\, \boldsymbol{M}_A\,[H] = AH - HA\,. \tag{6.8}$$

This formula was obtained by Mehrabadi and Nemat-Nasser (1987) using a different method.

Since $\boldsymbol{N}_A\,[H] = \boldsymbol{M}_A\,[AH - HA]$, by replacing H with $AH - HA$ in (6.8) we obtain the formula

$$\mathrm{I}_A\, \boldsymbol{N}_A\,[H] = A^2H + HA^2 - 2AHA\,. \tag{6.9}$$

Substitution of $(6.2)_1$ into (6.9) yields

$$\mathrm{I}_A \, \boldsymbol{N}_A \, [\boldsymbol{H}] = -\, 2\, \boldsymbol{AHA} + \mathrm{I}_A \, (\boldsymbol{AH} + \boldsymbol{HA}) - 2\, \mathrm{II}_A \, \boldsymbol{H} \, . \tag{6.10}$$

Recall that $\boldsymbol{X} = \boldsymbol{L}_A^{-1} \, [\boldsymbol{H}]$ is the solution of the tensor equations (4.2) and (4.3). As an application of the formula (6.10) for $\boldsymbol{N}_A \, [\boldsymbol{H}]$, observe that substitution of (6.10) into the identity $(4.5)_3$ yields the formulas

$$\begin{aligned} 2\, \mathrm{I}_A \, \boldsymbol{L}_A^{-1} \, [\boldsymbol{H}] &= \boldsymbol{AHA} + \mathrm{II}_A \, \boldsymbol{H} \\ &= 2\, \mathrm{II}_A \boldsymbol{H} + \mathrm{I}_{AH} \, \boldsymbol{A} - \mathrm{II}_A \, \mathrm{I}_H \, \boldsymbol{I} \, . \end{aligned} \tag{6.11}$$

The second formula follows from the first by Rivlin's identity (6.4). A formula equivalent to $(6.11)_1$ was obtained by Hoger (1986) using a different method; cf. equation (4.5) and the equation preceding (4.3) in her paper.

Finally, we derive some formulas for $\boldsymbol{L}_A \, [\boldsymbol{H}]$ which are valid for any tensor $\boldsymbol{H}$. Substitution of (6.10) into $(3.25)_2$ yields

$$2\, \mathrm{I}_A \, \boldsymbol{L}_A \, [\boldsymbol{H}] = \boldsymbol{H} + \mathrm{II}_A \, \boldsymbol{A}^{-1} \boldsymbol{H} \boldsymbol{A}^{-1} \, . \tag{6.12}$$

This also follows from $(6.11)_1$ with $\boldsymbol{A} \to \boldsymbol{A}^{-1}$ and the identitites $\mathrm{I}_{A^{-1}} = \mathrm{I}_A / \mathrm{II}_A$ and $\mathrm{II}_{A^{-1}} = 1/\mathrm{II}_A$. Recall that $\boldsymbol{A}^{-1}$ can be expressed in terms of $\boldsymbol{A}$ by the simple formula $(6.2)_2$. By substituting $(6.2)_2$ into (6.12) and expanding, we obtain

$$2\, \mathrm{I}_A \, \mathrm{II}_A \, \boldsymbol{L}_A \, [\boldsymbol{H}] = \boldsymbol{AHA} - \mathrm{I}_A \, (\boldsymbol{AH} + \boldsymbol{HA}) + (\mathrm{I}_A^2 + \mathrm{II}_A) \, \boldsymbol{H} \, . \tag{6.13}$$

Substitution of Rivlin's identities (6.3) and (6.4) into (6.13) yields

$$2\, \mathrm{I}_A \, \mathrm{II}_A \, \boldsymbol{L}_A \, [\boldsymbol{H}] = 2\, \mathrm{II}_A \boldsymbol{H} + \beta_1(\boldsymbol{A}, \boldsymbol{H}) \, \boldsymbol{A} + \beta_0(\boldsymbol{A}, \boldsymbol{H}) \, \boldsymbol{I} \, , \tag{6.14}$$

where β_1 is given by (6.5), and

$$\begin{aligned} \beta_0(\boldsymbol{A}, \boldsymbol{H}) &= \mathrm{tr} \, [\, ((\mathrm{I}_A^2 - \mathrm{II}_A) \, \boldsymbol{I} - \mathrm{I}_A \, \boldsymbol{A}) \, \boldsymbol{H}] \\ &= (\mathrm{I}_A^2 - \mathrm{II}_A) \, \mathrm{I}_H - \mathrm{I}_A \, \mathrm{I}_{AH} \, . \end{aligned} \tag{6.15}$$

The formulas (6.13) and (6.14) were obtained by Hoger and Carlson (1984).

7. AN APPLICATION TO THE KINEMATICS OF RIGID MOTIONS

The material in this section, with the exception of the last equation, is taken from § I.10 of the textbook by Truesdell (in press). Let W denote the spin tensor. Let M, E and J denote the rotational momentum, the Euler tensor and the tensor of inertia of a body relative to the (possibly time - dependent) point x_0. J and E are symmetric positive-definite tensors which are related as follows:

$$J = (\mathrm{tr}\, E)\, I - E \,. \tag{7.1}$$

If the body is undergoing a rigid motion, and if x_0 is either the location of the center of mass of the body or a fixed point of the body, then the rotational momentum relative to x_0 is related to the spin tensor and the Euler tensor relative to x_0 by the formula

$$M = -\,(EW + WE) \,. \tag{7.2}$$

Truesdell (in press) utilized Guo's formula (see (5.18)) to solve (7.2) for the spin tensor in terms of the rotational momentum and the Euler tensor:

$$W = \frac{-1}{\mathrm{I}_E\, \mathrm{II}_E - \mathrm{III}_E} \left[\left(\mathrm{I}_E^2 - \mathrm{II}_E \right) M - \left(E^2 M + M E^2 \right) \right] . \tag{7.3}$$

The purpose of this section is to point out that W has a simpler expression in terms of the tensor of inertia. Indeed, since the solution of (7.2) is $W = L_E\,[-M]$, by setting $A = E$ and $G = -M$ in (5.4), (5.5) and (5.16) and using (7.1), we find that

$$W = -\,\frac{1}{\mathrm{III}_J}\, JMJ \,. \tag{7.4}$$

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